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Systematic perturbation solution for Brownian motion in a biased periodic potential field

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In this paper, a systematic inverse friction expansion treatment of the mobility of a Brownian particle moving under a periodic potential and a constant bias is presented by considering the stationary solution of the Fokker-Planck equation. By solving different orders of truncations of the infinite hierarchy of equations equivalent to the Fokker-Planck equation and solving the infinite tridiagonal matrix, respectively, explicit integral and series solutions are obtained.

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I. INTRODUCTION

The motion of a Brownian particle in a periodic potential under an external constant force has been widely investigated to explain the behavior of a variety of systems of practical importance, such as the damped pendulum [3], superionic conductor [4–8], Josephson tunneling junction [9], phase locked loop [10], rotation of dipoles driven by a constant field [11–15], and so on. One of the main tasks in treating all these problems is the calculation of the averaged velocity or the mobility.

Let us consider the following Langevin equation:

$$m\ddot{x}+\gamma\dot{x}+f'(x)=F+\Gamma(t)\ , \tag{1.1}$$

where m,γ,F are the mass of the particle, friction coefficient, and the constant external force, respectively, $f(x)$ denotes the potential and $-f'(x)=-df(x)/dx$ represents the periodic force. The noise term $\Gamma(t)$ is white in accordance with the assumption of a friction proportional to the velocity,

$$\langle\Gamma(t)\rangle=0,\quad \langle\Gamma(t)\Gamma(t')\rangle=2\gamma\vartheta\delta(t-t')\ , \tag{1.2}$$

where $\vartheta=k_BT$ is the thermal power and k_B the Boltzmann constant. With a suitable scaling, Eq. (1.1) can be transformed to

$$\ddot{x}+\gamma\dot{x}+f'(x)=F+\Gamma(t)\ , \tag{1.3}$$

where $f(x)=f(x+2\pi)$.

Equation (1.3) corresponds to a Fokker-Planck equation (FPE),

$$\frac{\partial P(x,v,t)}{\partial t}+\frac{\partial}{\partial x}[vP(x,v,t)]+\frac{\partial}{\partial v}\{[F-f'(x)]P(x,v,t)\}=\gamma\frac{\partial}{\partial v}\left[vP+\vartheta\frac{\partial P}{\partial v}\right]\ , \tag{1.4}$$

with $v=\dot{x}=dx/dt$ being the velocity.

Equation (1.4) can be simplified in various approximations. In the large friction limit, the thermal relaxation time is much shorter than the spatial diffusion time, thus the two-dimensional FPE can be reduced to the Smoluchowski equation,

$$\frac{\partial P(x,t)}{\partial t}=\frac{1}{\gamma}\frac{\partial}{\partial x}[f'(x)-F]P(x,t)+\frac{\vartheta}{\gamma}\frac{\partial^2P(x,t)}{\partial x^2}\ , \tag{1.5}$$

where $P(x,t)=\int_{-\infty}^{+\infty}P(x,v,t)dv$ is the distribution function for x. The stationary mobility of (1.5) can be easily evaluated [1], i.e.,

$$\mu=\frac{2\pi}{F}\frac{1-\exp(-2\pi F/\vartheta)}{\Omega\Delta-[1-\exp(-2\pi F/\vartheta)]\Lambda}\ , \tag{1.6}$$

with u,v,w given by

$$\Omega=\gamma\int_0^{2\pi}\exp[f(x)-Fx]dx\ , \tag{1.7}$$

$$\Delta=\int_0^{2\pi}\exp[-f(x)+Fx]dx\ , \tag{1.8}$$

$$\Lambda=\gamma\int_0^{2\pi}dx\,\exp[-f(x)+Fx]\int_0^{x}\exp[f(\xi)-F\xi]d\xi\ . \tag{1.9}$$

In the low friction case, x and v become fast variables, and the energy $E=v^2/2+f(x)$ or the action $I(E)=(2\pi)^{-1}\oint\dot{x}\,dx$ becomes a slow variable, again a one-dimensional FPE can be obtained after an adiabatic elimination treatment [1]. Analytic results have been obtained in this limiting case. Risken and co-workers [1,2] have applied the matrix-continued-fraction method (MCFM) to Eq. (1.4) and investigated the behavior of the system for intermediate frictions by numerical computation. Rather few explicit expressions have been derived to date for this intermediate range. The system (1.3) [or equivalently, (1.4)] is a rather old model. However, further investigation of this model is still significant if it brings some new results, because this model has extremely wide applications and is very typical in noise and transportation problems.

In this paper, we will present a systematic and explicit solution in inverse friction expansion on the basis of

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Risken's MCFM. This paper is arranged as follows. Section II gives a brief review of the MCFM as the starting point of our analysis. In Sec. III, an integral treatment for a general smooth periodic potential is given. The results of the first two orders are in agreement with those obtained in Refs. [1,2]. In Sec. IV, we discuss a particular potential $f(x)=-d\cos x$ by applying an eigenvalue expansion approach, and a series form of the solution is provided. It is shown that both forms (the integral form and the series form) are equivalent.

II. THE MCFM

In this paper, we discuss the stationary case, i.e.,

$$\partial P(x,v,t)/\partial t=0 \ , \tag{2.1}$$

where $P(x,v)$ is a periodic distribution with respect to x. By expanding $P(x,v)$ in terms of $\Psi_n(v)$, we have

$$P(x,v)=\Psi_0(v)\sum_{n=0}^{\infty} C_n(x)\Psi_n(v) \ , \tag{2.2}$$

where $\Psi_n(v)=H_n(v/\sqrt{2\vartheta})\exp(-v^2/4\vartheta)/(n!2^2\sqrt{2\pi\vartheta})^{1/2}$, and the $H_n(x)$ are the Hermite polynomials [1], a hierarchy equivalent to the FPE (1.4) can be given by

$$\sqrt{n}K^-C_{n-1}+n\gamma C_n+\sqrt{n+1}K^+C_{n+1}=0 \ , \tag{2.3}$$

where $n=0,1,2,\ldots,\infty$, and k^+,K^- read

$$K^+=\sqrt{\vartheta}\partial/\partial x \ , \tag{2.4a}$$

$$K^-=K^++\vartheta^{-1/2}[f'(x)-F] \ . \tag{2.4b}$$

Evidently, $C_0(x)=\int_{-\infty}^{+\infty}P(x,v)dv$ is the reduced distribution with respect to x, and $C_1(x)=C$ is a constant. It can be easily obtained that

$$\langle v\rangle=2\pi\sqrt{\vartheta}C \ , \tag{2.5}$$

and the mobility is

$$\mu=\langle v\rangle/F=\sqrt{\vartheta}\frac{2\pi}{F}C \ . \tag{2.6}$$

A direct computation leads to [1]

$$C_0=MC_1 \ , \tag{2.7}$$

where M is an operator-continued fraction, i.e.,

$$M=-\gamma(K^-)^{-1}\left[I-\frac{1}{\gamma^2}K^+\cfrac{1}{I-\cfrac{1}{2\gamma^2}K^+\cfrac{1}{I-\cfrac{1}{3\gamma^2}K^+\cdots K^-}K^-}K^-\right] , \tag{2.8}$$

where $(K^-)^{-1}$ denotes the inversion operator of K^- and I is the unit operator. We can further expand $C_n(x)$ in terms of the Fourier series,

$$C_n(x)=\sum_{\alpha=-\infty}^{\infty} C_n^{\alpha}e^{i\alpha x} \ , \tag{2.9}$$

then $C_n(x)$ corresponds to a vector $(\cdots C_n^{-\alpha}\cdots C_n^0\cdots C_n^{\alpha}\cdots)$, and M now is a matrix-continued fraction, where $K^{\pm}$ are matrices with the elements

$$(K^+)^{\alpha\beta}=i\sqrt{\vartheta}\alpha\delta_{\alpha\beta} \ , \tag{2.10a}$$

$$(K^-)^{\alpha\beta}=(K^+)^{\alpha\beta}-\vartheta^{-1/2}F\delta_{\alpha\beta}+\vartheta^{-1/2}[f'(x)]^{\alpha\beta} \ , \tag{2.10b}$$

$$[f'(x)]^{\alpha\beta}=(2\pi)^{-1}\int_0^{2\pi}f'(x)e^{i(\beta-\alpha)x}dx \ . \tag{2.10c}$$

An explicit expression may be derived from the normalization as

$$\mu=\vartheta^{1/2}F^{-1}[M^{00}]^{-1} \ , \tag{2.11}$$

where M^{ij} is the element of M.

The above methods and expressions have been presented and used for numerical computation by Risken and co-worker [1,2]. Here, we take these results as the starting point for our analytic study.

III. THE PERTURBATIVE INTEGRAL EXPRESSION OF THE MOBILITY

By writing M in the form

$$M=-\gamma(K^-)^{-1}H \ , \tag{3.1}$$

Eq. (2.7) is transformed to

$$K^-C_0(x)=-\gamma HC_1(x) \ . \tag{3.2}$$

From now on, we normalize $\vartheta=k_BT=1$ for simplicity. Recalling $C_1(x)=C$ is constant and using

$$q(x)=\frac{1}{C}\gamma HC_1(x) \ , \tag{3.3}$$

we obtain a simple differential equation for the variation of the reduced distribution function of x,

$$dC_0(x)/dx+[f'(x)-F]C_0(x)=-Cq(x) \ . \tag{3.4}$$

The solution is expressed by

$$C_0(x)=Ke^{-f(x)+Fx}-Ce^{-f(x)+Fx}\int_0^x q(\xi)e^{f(\xi)-F\xi}d\xi \ . \tag{3.5}$$

The constants K and C may be derived in terms of the normalization condition and periodic condition of $C_1(x)$, i.e.,

$$K/C = e^{2\pi F}\Omega/(e^{2\pi F}-1) , \tag{3.6a}$$

$$K\Delta - C\Lambda = 1 , \tag{3.6b}$$

where Ω, Δ, Λ are

$$\Omega = \int_0^{2\pi} q(x) e^{f(x)-Fx} dx , \tag{3.7a}$$

$$\Delta = \int_0^{2\pi} e^{-f(x)+Fx} dx , \tag{3.7b}$$

$$\Lambda = \int_0^{2\pi} dx\, e^{-f(x)+Fx} \int_0^x q(\xi) e^{f(\xi)-F\xi} d\xi . \tag{3.7c}$$

Finally we obtain the mobility,

$$\mu = \frac{2\pi}{F} \frac{(1-e^{-2\pi F})}{\Omega\Delta - (1-e^{-2\pi F})\Lambda} . \tag{3.8}$$

Then, we have an analytic expression for the mobility valid for arbitrary friction strength. Now the structure of the solution of the mobility for arbitrary friction is clear, it has exactly the same form as the overdamped result Eq. (1.6), as it is posed in terms of three quantities that can be computed by means of formulas very similar to (1.7)–(1.9), i.e., to compute to any order single and double integrals with integrands given explicitly, just as in Eq. (1.6). The only difference from the overdamped case is that in (3.8) a single renormalization function $q(x)$ should be considered which reduces to $q(x) = \gamma$ in the overdamped case.

The remaining work is focused on the calculation of the single kernel function $q(x)$, which can be systematically derived by an inverse friction expansion. Because the MCFM is valid for finite γ, the expansion can be executed to any order. We have to the first order

$$q(x) = \gamma .$$

To order γ^{-1}, we have

$$q(x) = \gamma \left[I - \frac{1}{\gamma^2} K^+ K^- \right] C_1(x)/C ,$$

leading to

$$q(x) = \gamma - \frac{1}{\gamma} f^{(2)}(x) , \tag{3.9}$$

where $f^{(n)}(x) = d^n f(x)/dx^n$.

The expressions $q(x) = \gamma$ and (3.9) are in agreement with the known results in Refs. [1] and [2]. However, we can now proceed to higher orders, yielding the same expression for the mobility as (3.8), with renormalization $q(x)$ which is also easily obtained to any finite order. To the next order γ^{-3}, we have

$$q(x) = \gamma \left\{ I - \frac{1}{\gamma^2} K^+ K^- - \frac{1}{2\gamma^4} (K^+)^2 (K^-)^2 \right\} C_1(x)/C ,$$

leading to

$$\begin{aligned} q(x) = \gamma &- \frac{1}{\gamma} f^{(2)}(x) \\ &- \frac{1}{2\gamma^3} \{ f^{(4)}(x) + 2[f^{(1)}(x) - F] f^{(3)}(x) \\ &\quad + 2[f^{(2)}(x)]^2 \} . \end{aligned} \tag{3.10}$$

There is no difficulty in systematically representing $q(x)$ in higher orders by computing derivatives only.

IV. THE MOBILITY FOR COSINE CASE: A SERIES EXPRESSION

For the cosine potential,

$$f(x) = -d \cos x , \tag{4.1}$$

the matrix K^- becomes a tridiagonal one,

$$(K^-)^{\alpha\beta} = \vartheta^{-1/2} \left\{ (i\alpha\vartheta - F)\delta_{\alpha,\beta} + \frac{id}{2}\delta_{\alpha,\beta-1} - \frac{id}{2}\delta_{\alpha,\beta+1} \right\} . \tag{4.2}$$

In Eq. (3.1), the matrix-continued fraction H may be expanded with respect to γ^{-1}. The first few terms are written as

$$H = H_0 + H_1 + H_2 + H_3 + H_4 + \cdots , \tag{4.3a}$$

$$H_0 = I , \tag{4.3b}$$

$$H_1 = -\frac{1}{\gamma^2} K^+ K^- , \tag{4.3c}$$

$$H_2 = -\frac{1}{2!\gamma^4} (K^+)^2 (K^-)^2 , \tag{4.3d}$$

$$H_3 = -\frac{1}{3!\gamma^6} (K^+)^3 (K^-)^3 - \frac{1}{2^2\gamma^6} (K^+)^2 (K^- K^+)(K^-)^2 , \tag{4.3e}$$

$$\begin{aligned} H_4 = &-\frac{1}{4!\gamma^8} (K^+)^4 (K^-)^4 - \frac{1}{2\times 3^2 \gamma^8} (K^+)(K^- K^+)(K^-) \\ &- \frac{1}{2^3\gamma^8} (K^+)(K^- K^+)^2 (K^-)^2 \\ &- \frac{1}{2\times 2\times 3\gamma^8} (K^+)^2 (K^- K^+)(K^+ K^-)(K^-)^2 \\ &- \frac{1}{2\times 2\times 3\gamma^8} (K^+)^2 (K^+ K^-)(K^- K^+)(K^-)^2 . \end{aligned} \tag{4.3f}$$

Now the main difficulty in computing (3.1) lies in evaluating the inverse $(K^-)^{-1}$. In the appendix, the analytic and explicit solution for the inverse of the infinite tridiagonal matrix K^- is given in a completely closed form. The result reads

$$\begin{aligned} [(K^-)^{-1}]^{kj} = &-\sum_{n=-\infty}^{\infty} (-1)^{k+n} \frac{\vartheta^{1/2}}{F/\vartheta + in} \\ &\times I_{k+n}(d/\vartheta) I_{j+n}(d/\vartheta) , \end{aligned} \tag{4.4}$$

where i denotes the imaginary unit and $I_n(x)$ is the modified Bessel function. Considering both Eqs. (4.3) and (4.4), we can reproduce the solution of M to any order in the inverse γ expansion. Therefore, we can expand the mobility in the form

$$\mu = \left\{ \sum_{L=0}^{\infty} \left[\frac{A_L}{\gamma^{2L-1}} \right] \right\}^{-1} , \tag{4.5}$$

where the factor A_L reads

$$A_L=-F\vartheta^{-1/2}\sum_{l=-L}^{L}\left\{a_L^l[(K^-)^{-1}]^{0l}\right\}$$
$$=\begin{cases}-F\vartheta^{-1/2}[(K^-)^{-1}]^{00} & \text{for } L=0;\\ -2F\vartheta^{-1/2}\sum_{l=1}^{L}\mathrm{Re}\left\{a_L^l[(K^-)^{-1}]^{0l}\right\} & \text{for } L>0\ .\end{cases} \tag{4.6}$$

Here Re takes the real part, a_L^l are given by the multiplication of the matrices K^+ and K^-. The first few coefficients are given as follows:

$$L=0,\quad a_0^0=1\ ,$$
$$L=1,\quad a_1^{-1}=\overline{[a_1^1]}=d/2\ ,$$
$$L=2,\quad a_2^{-1}=\overline{[a_2^1]}=-d(\vartheta-2iF)/4\ ,$$
$$a_2^{-2}=\overline{[a_2^2]}=-d^2/2\ ,$$
$$L=3,\quad a_3^{-1}=\overline{[a_3^2]}=-d(12F^2-5\vartheta^2+3d^3+15iF\vartheta)/24\ ,$$
$$a_3^{-2}=\overline{[a_3^2]}=d^2(9\vartheta-8iF)/4\ ,$$
$$a_3^{-3}=\overline{[a_3^3]}=9d^3/8\ ,$$
$$L=4,\quad a_4^{-1}=\overline{[a_4^1]}=\frac{d}{2}[\vartheta(17F^2/8-7\vartheta^2/18+17d^2/16)$$
$$-iF(F^2-14\vartheta^2/9+d^2)]\ ,$$
$$a_4^{-2}=\overline{[a_4^2]}=d^2(11F^2/2+d^2-659\vartheta^2/72$$
$$+111iF\vartheta/8),$$
$$a_4^3=\overline{[a_4^3]}=-9d^3(57\vartheta-32iF)/32\ ,$$
$$a_4^{-4}=\overline{[a_4^4]}=-7d^4/2\ , \tag{4.7}$$

where $\overline{[a_L^l]}$ denotes the complex conjugate of a_L^l. Higher order coefficients a_L^l can also be successively obtained. It can be verified that the results (4.5) and (3.8) are equivalent for the cosine potential. Both solutions have their own advantages. Equation (3.8) is valid for arbitrary periodic potentials while Eq. (4.5) is valid only for the cosine potential. The series expression for a system with an arbitrary periodic potential may be rather complicated. On the other hand, the series form of (4.5) is easier for numerical computation than that of (3.8). In the next section, we present numerical results based on Eq. (4.5).

V. NUMERICAL RESULTS

Taking the widely used cosine potential case as an example, we show, numerically, the results obtained in Sec. III and IV. The mobility may also be expressed in the following form [2] :

$$\mu=B_0\gamma^{-1}-B_1\gamma^{-3}-B_2\gamma^{-5}-B_3\gamma^{-7}-B_4\gamma^{-9}-\cdots \tag{5.1}$$

with

$$\begin{aligned}&B_0=1/A_0,\quad B_1=A_1/A_0^2\ ,\\ &B_2=(A_2A_0-A_1^2)/A_0^3\ ,\\ &B_3=(A_3A_0^2-2A_2A_1A_0+A_1^3)/A_0^4\ ,\\ &B_4=(A_4A_0^3-2A_3A_1A_0^2\\ &\qquad+3A_2A_1^2A_0-A_2^2A_0^2-A_1^4)/A_0^5\ldots\ ,\end{aligned} \tag{5.2}$$

where A_i are given in (4.6) and (4.7).

We plot $\gamma\mu$ to the first three orders against the external force F with $d=1.0$ and $\vartheta=1$ in Fig. 1 ($\gamma=5.0$) and Fig. 2 ($\gamma=2.0$). We find that for $\gamma=5.0$, the first order approximation is already very good and other orders give results without visible differences. In Fig. 2, for smaller γ, the corrections of higher orders become important, especially in the larger F region. However, even for not too large γ, the analytic results obtained in Sec. III and Sec. IV are also useful, and the first few terms may already be enough to produce rather accurate results.

In Figs. 3–5, we plot the first three coefficients B_i against the barrier height d at $\vartheta=1.0$ and $F=4.0, 3.0, 2.0, 1.0$, and 0.5, respectively. In Fig. 3, B_0 decreases to 0 as d increases, indicating that particles escape less easily from a deep potential well. An interesting point is that there is a peak for B_1 and a valley for B_2 for the response curves, as shown in Figs. 4 and 5. This indicates complicated behavior of the mobility $\gamma\mu$ at moderate and small γ due to the competition of the periodic potential (d), external force (F), and random noise (temperature ϑ). However, the detailed features and their mechanism, and the radius of convergence of the expansion are still unknown.

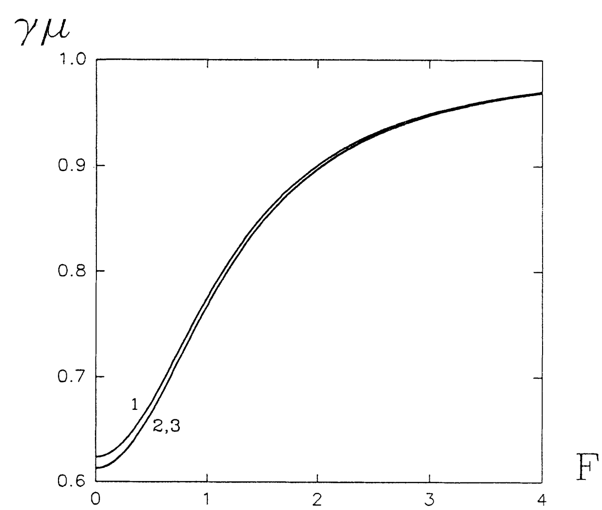

FIG. 1. Mobility multiplied by the friction $\gamma\mu$ vs the external force F. $\gamma=5.0$, $d=\vartheta=1.0$, and the curves 1, 2, and 3 represent the approximations up to orders γ, γ^{-1}, and γ^{-3}, respectively. The good fit to the first order approximation is obvious. Curves 2 and 3 cannot be distinguished. The quantities in this figure and all the following figures are dimensionless.

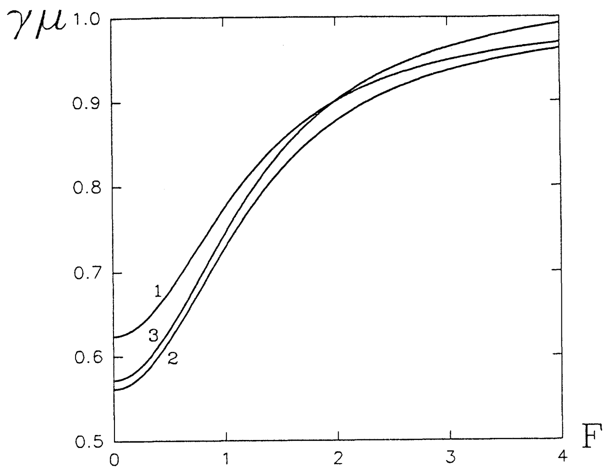

FIG. 2. $\gamma\mu$ vs F, $\gamma=2.0$, and $d=\vartheta=1.0$. Curves 1, 2, and 3 have the same meanings as in Fig. 1. The corrections of higher orders become important.

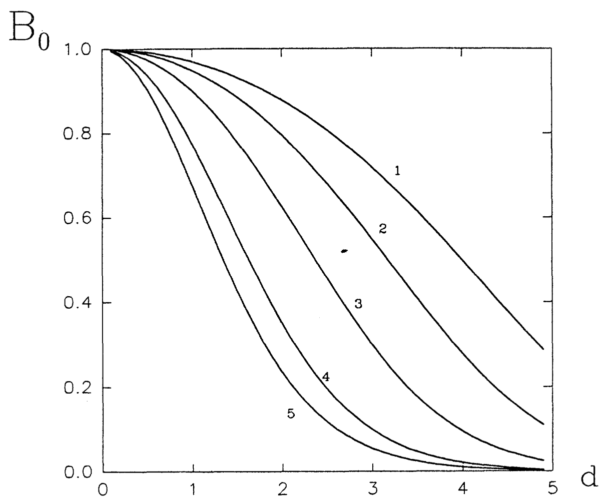

FIG. 3. The expansion coefficient B_0 plotted against the barrier height d. $\vartheta=1.0$, curves 1, 2, 3, 4, and 5 correspond to $F=4.0$, 3.0, 2.0, 1.0, and 0.5, respectively.

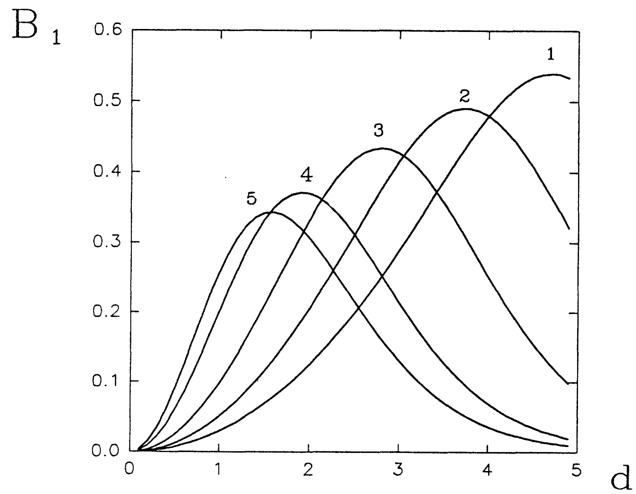

FIG. 4. The same as Fig. 3 with B_0 replaced by B_1.

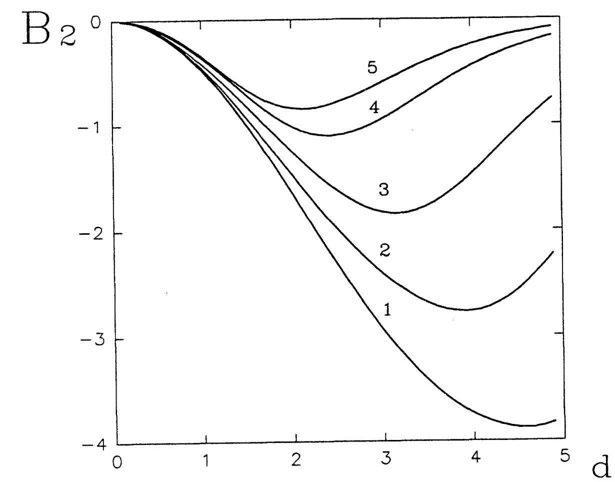

FIG. 5. The same as Fig. 3 with B_0 replaced by B_2.

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APPENDIX

In order to compute the inverse of the matrix,

$$(K^-)^{pq}=(ipa-b)\delta_{pq}+ih\delta_{p,q+1}-ih\delta_{p,q-1}\ , \tag{A1}$$

we need to know the eigenvalues and eigenfunctions of K^-. Generally, K^- is not Hermitian, so we should find the biorthogonal set of K^-. We write the right eigenfunctions in the form

$$|n\rangle=(\cdots c_n^{-q}\cdots c_n^{0}\cdots c_n^{q}\cdots)^T\ , \tag{A2}$$

and the left ones

$$\langle n|=(\cdots a_n^{-q}\cdots a_n^{0}\cdots a_n^{q}\cdots)\ . \tag{A3}$$

Then the eigenequations read

$$K^-|n\rangle=\lambda_n|n\rangle\ , \tag{A4a}$$

$$\langle n|K^-=\lambda_n\langle n|\ , \tag{A4b}$$

with λ_n being the corresponding eigenvalues.

First, we solve the right eigenfunctions. For the q-th component, we have

$$(iaq-b-\lambda_n)c_n^q+ihc_n^{q+1}-ihc_n^{q-1}=0\ , \tag{A5}$$

where $a=\vartheta^{1/2}$, $b=F\vartheta^{-1/2}$, and $h=-0.5d\vartheta^{-1/2}$. Introducing the generating function

$$g_n^R(s)=\sum_{q=-\infty}^{\infty}c_n^q s^q\ , \tag{A6}$$

we obtain a partial differential equation

$$ias\frac{dg_n^R}{ds}-(\lambda_n+b)g_n^R+i(hs^{-1}-hs)g_n^R=0\ , \tag{A7}$$

which has the solution

$$g_n^R(s)=e^{h/a(s+1/s)}s^{(\lambda_n+b)/ia} . \tag{A8}$$

A complete set of eigenvectors can be constructed by setting $(\lambda_n+b)/ia$ to integer numbers, which leads to

$$\lambda_n=-b+ina , \tag{A9}$$

where $n=-\infty,\ldots,1,0,1,\ldots,\infty$. Thus $|n\rangle$ can be obtained such that the component c_n^q is identified to the coefficient of s^q in the power expansion of $g_n^R(s)$. In fact, c_n^q can be provided directly by expanding

$$e^{h/a(s+1/s)}=\sum_{n=-\infty}^{\infty} c_n s^n , \tag{A10}$$

and identifying c_n^q to c_{q-n}.

The same procedure may be applied to the left eigenfunction, and finally we obtain

$$g_n^L(s)=e^{-h/a(s+1/s)}s^n . \tag{A11}$$

Similar discussion also leads to

$$a_n^q=\tilde{a}_{q-n} , \tag{A12}$$

where $\tilde{a}_n$ is the expansion coefficient of $e^{-h/a(s+1/s)}$ with respect to s^n.

Thus $|n\rangle,\langle n|$ form a biorthogonal set. The function $e^{x(s+1/s)}$ is the generating function of the modified Bessel function $I_n(x)$, i.e.,

$$e^{x(s+1/s)}=\sum_{n=-\infty}^{\infty} I_n(x)s^n , \tag{A13}$$

where $I_n(x)=\sum_{k=0}^{\infty}[1/k!\Gamma(k+n+1)](x/2)^{n+2k}$, and $\Gamma(k+n+1)=(n+k)!$, the γ function. It can be verified that

$$I_0^2(x)+2\sum_{n=1}^{\infty}(-1)^n I_n^2(x)=1 . \tag{A14}$$

Therefore, the normalization condition $\sum_{n=-\infty}^{\infty}a_n c_n=1$ is satisfied and the biorthogonal relation of eigenvalues $|m\rangle$ and $\langle m|$ may be expressed by

$$\langle n|m\rangle=\sum_{n=-\infty}^{\infty} c_m a_n=\delta_{mn}, . \tag{A15}$$

Any infinite matrix can be expanded on the basis of the biorthogonal set $|n\rangle$ and $\langle n|$. For K^- and $(K^-)^{-1}$, we have

$$K^-=\sum_{n=-\infty}^{\infty}\lambda_n|n\rangle\langle n| , \tag{A16}$$

$$(K^-)^{-1}=\sum_{n=-\infty}^{\infty}\frac{1}{\lambda_n}|n\rangle\langle n| . \tag{A17}$$

Therefore, the element of (A17) is

$$[(K^-)^{-1}]^{pq}=\sum_{n=-\infty}^{\infty}\frac{1}{\lambda_n}\langle p|n\rangle\langle n|q\rangle . \tag{A18}$$

In terms of (A9) and (A13), we get, finally,

$$(K^{-1})^{pq}=\sum_{n=-\infty}^{\infty}\frac{1}{-b+ina}I_{p-n}(2h/a)I_{q-n}(-2h/a) , \tag{A19}$$

leading to Eq. (4.4). It can be verified that

$$\overline{(K^{-1})^{pq}}=(K^{-1})^{-p,-q} , \tag{A20}$$

where the bar on the top indicates the imaginary conjugation.

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